HOMOTHETICITY AND HOMOGENEITY
(Demonstrating the contrary)

Homogeneity

West Germany:

$$Y = \alpha_0 + \alpha_L L + \frac{1}{2}\beta_L L^2 + \frac{1}{2}\beta_K K^2 + \beta_{LK} LK^2$$

If f(L, K) is homogeneous, then Euler's theorem[1] applies -- i.e.,

$$r \cdot Y \;=\; f_L L + f_K K$$

such that $$f(tL,\, tK) \;=\; t^r \cdot f(L, K)$$

Consider now the two marginal product functions

$$f_L \;=\; \alpha_L + \beta_L L + \beta_{LK} K$$

and $$f_K \;=\; \beta_K K + \beta_{LK} L$$

Substituting these above shows

$$\begin{aligned} r \cdot Y \;&=\; f_L L + f_K K \\ &=\; \alpha_L + \beta_L L^2 + 2\beta_{LK} LK + \beta_K K^2 \end{aligned}$$

Splitting terms and adding zero obtains:

$$r \cdot Y \;=\; 2Y - 2\alpha_0 - \alpha_L L$$

or

$$r \;=\; 2 - \frac{2\alpha_0 + \alpha_L L}{Y}$$

Only if Y and $\frac{L}{Y}$ are constant could f(L, K) be homogeneous. Clearly this is unlikely; moreover, the data shows that this is not true.

France:

$$Y^* = A_L L^* + \frac{1}{2} B_L L^{*2} + B_{LK} L^* K^*$$

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[1]Silberberg, 1978: pp. 84-93.

$$g_L = A_L + B_L L^* + B_{LK} K^*$$

$$g_K = B_{LK} L^*$$

Substituting into the Euler expression obtains:

$$r \cdot Y^* = f_L L^* + f_K K^* = A_L L^* + B_L L^{*2} + 2B_{LK} L^* K^*$$

Splitting terms and adding zero where appropriate the above expression can be written as follows:

$$r \cdot Y^* = 2 \; Y^* - A_L L^*$$

or

$$r = 2 - A_L \frac{L^*}{Y^*}$$

Only if $\frac{L^*}{Y^*}$ were constant could g(L*, K*) be homogeneous.

<u>Homotheticity</u>

West Germany:

Is the West German function homothetic? Consider the following:

$$Y = \alpha_0 + \alpha_L L + \frac{1}{2} \beta_L L^2 + \frac{1}{2} \beta_K K^2 + \beta_{LK} LK^2$$

where $$f_L = \alpha_L + \beta_L L + \beta_{LK} K$$

and $$f_K = \beta_K K + \beta_{LK} L$$

For constant Y we obtain

$$-\frac{f_L}{f_K} = \frac{\delta K}{\delta L} = -\frac{\alpha_L + \beta_L L + \beta_{LK} K}{\beta_K K + \beta_{LK} L}$$

If the function is homothetic then by Euler's theorems the following must hold:

$$\frac{\alpha_L + \beta_L L + \beta_{LK} K}{\beta_K K + \beta_{LK} L} = \frac{\alpha_L + t\,(\beta_L L + \beta_{LK} K)}{t\,(\beta_K K + \beta_{LK} L)}$$

This is clearly not the case.

France:

From our French production function we obtain:

$$Y^* = A_L L^* + \frac{1}{2} B_L L^{*2} + B_{LK} L^* K^*$$

where $$g_L = A_L + B_L L^* + B_{LK} K^*$$

and $$g_K = B_{LK} L^*$$

If our function is homothetic then by Euler's theorems the following must hold:

$$-\frac{g_L}{g_K} = \frac{\delta K^*}{\delta L^*} = -\frac{A_L + B_L L^* + B_{LK} K^*}{B_{LK} L^*}$$

Clearly,

$$-\frac{A_L + B_L L^* + B_{LK} K^*}{B_{LK} L^*} \neq -\frac{A_L + t\,(B_L L^* + B_{LK} K^*)}{t\,B_{LK} L^*}$$

Thus these functions are neither homogeneous, nor homothetic.

COMPARATIVE STATICS

Pre-Trade Algebra (West Germany)

$$Y = \alpha_0 + \alpha_L L + \frac{1}{2}\beta_L L^2 + \frac{1}{2}\beta_K K^2 + \beta_{LK} LK$$

Marginal $\quad \frac{\delta Y}{\delta L} = \alpha_L + \beta_L L + \beta_{LK} K$

and $\quad \frac{\delta Y^2}{\delta L^2} = \beta_L$

Cross $\quad \frac{\delta Y}{\delta K} = \beta_K K + \beta_{LK} L$

Marginal $\quad \frac{\delta Y^2}{\delta L^2} = \beta_K$

Products $\quad \frac{\delta Y^2}{\delta L^2 \delta K^2} = \beta_{LK}$

Pre-Trade Algebra (France)

$$Y^* = A_L L^* + \frac{1}{2} B_L L^{*2} + B_{LK} L^* K^*$$

Marginal $\quad \frac{\delta Y^*}{\delta L^*} = A_L + B_L L^* + B_{LK} K^*$

and $\quad \frac{\delta Y^{*2}}{\delta L^{*2}} = B_L$

Cross $\quad \frac{\delta Y^*}{\delta K^*} = B_{LK} L^*$

Marginal $\quad \frac{\delta Y^{*2}}{\delta K^{*2}} = B_K$

Products $\quad \frac{\delta Y^{*2}}{\delta L^*{}_2 \delta K^{*2}} = B_{LK}$

SCENARIO I

Full Employment Conditions:

$$\bar{L} = L + L^*$$
$$K = K$$
$$K^* = K^*$$

Equilibrium Marginal Products:

<u>Labor</u>

$$mp_L = mp_L^* \cdot \rho_Y$$

$$\alpha_L + \beta_L L + \beta_{LK} K = \left(A_L + B_L L^* + B_{LK} K^*\right)\rho_Y$$

Substituting for $L^* = \bar{L} - L$

obtains

$$\alpha_L + \beta_L L + \beta_{LK} K = \left(A_L + B_L(\bar{L} - L) + B_{LK} K^*\right)\rho_Y$$

Solving explicitly for L yields the post-trade equilibrium value for West German labor, L_1:

$$L_1 = \left(\frac{1}{\beta_L + \rho_Y B_L}\right) \cdot \left(\left(A_L + B\bar{L} + B_{LK} K^*\right)\rho_Y - \alpha_L - \beta_{LK} K\right)$$

Functionally this value can be shown as:

$$L^1 = L^1(\bar{L}, K, K^*, \rho_Y)$$

Substituting the new post-trade equilibrium level for West German labor into the full employment condition obtains the post-trade French equilibrium labor level, L_1^*:

$$L_1^* = \bar{L} - L_1^*$$

Functionally this can be written as:

$$L_1^*(\bar{L}, K, K^*, \rho_Y) = \bar{L} - L_1^*(\bar{L}, K, K^*, \rho_Y)$$

Substituting L_1 and L_1^* into the marginal product functions for labor obtains the post-trade values:

$$mp_L = \alpha_L + \beta_L L_1 + \beta_{LK} K$$

$$mp_L^* = A_L + B_L L_1^* + B_{LK} K^*$$

<u>Capital Stock</u>

The pre-trade marginal products for West German and French capital are known to be:

$$mp_K = \beta_K K + \beta_{LK} L$$

and $$mp_K^* = B_{LK} L^*$$

Substituting the post trade equilibrium values for labor obtains the post-trade marginal products for capital:

$$mp_K = \beta_K K + \beta_{LK} L_1$$
$$mp_K^* = B_{LK} L_1^*$$

SCENARIO II

Full Employment Conditions: $$\bar{L} = L + L^*$$

$$\bar{K} = K + K^* \rho_K$$

Equilibrium Marginal Products:

<u>Labor</u>: $$mp_L = mp_L^*$$

$$\alpha_L + \beta_L L + \beta_{LK} K = \left(A_L + B_L L^* + B_{LK} K^*\right) \rho_Y$$

Substituting for $$L^* = \bar{L} - L$$

and $$K^* = (\bar{K} - K)\frac{1}{\rho_K}$$

obtains

$$\alpha_L + \beta_L L + \beta_{LK} K = \left(A_L + B_L(\bar{L} - L) + B_{LK}(\bar{K} - K)\frac{1}{\rho_K}\right) r_Y$$

Solving for L explicitly obtains L =

$$= \left(\frac{1}{\beta_L + \rho_Y B_L}\right)\left(\left(A_L + B_L \bar{L} + B_{LK}(\bar{K} - K)\frac{\rho_Y}{\rho_K}\right)\rho_Y - \alpha_L - \beta_{LK} K\right)$$

<u>Capital Stock</u>: $mp_K = mp_K^* \cdot \rho_Y$

$$\beta_K K + \beta_{LK} L = B_{LK} L^* \rho_Y$$

Substituting for $L^* = \bar{L} - L$

obtains: $\beta_K K + \beta_{LK} L = B_{LK}(\bar{L} - L)\rho_Y$

Solving explicitly for K yields:

$$K = \frac{1}{\beta_K}\left(B_{LK}(\bar{L} - L)\rho_Y - \beta_{LK} L\right)$$

Having solved for L and K explicitly the SAS procedure SYSNLIN was run to solve simultaneously for the equilibrium post-trade values for West Germany, L_2 and K_2.[1] From these the equilibrium post-trade values for France, L_2^* and K_2^* were obtained.

$$L_2 = L_2(\bar{L}, \bar{K}, \rho_K, \rho_Y)$$

$$K_2 = K_2(\bar{L}, \bar{K}, \rho_K, \rho_Y)$$

and

$$L_2^* = L_2^*(\bar{L}, \bar{K}, \rho_K, \rho_Y) = \bar{L} - L_2(\bar{L}, \bar{K}, \rho_K, \rho_Y)$$

$$K_2^* = K_2^*(\bar{L}, \bar{K}, \rho_K, \rho_Y) = \bar{K} - K_2(\bar{L}, \bar{K}, \rho_K, \rho_Y)$$

The post-trade marginal products can be found by substituting the post trade equilibrium values for labor and capital:

<u>Labor</u> $mp_L = \alpha_L + \beta_L L_2 + \beta_{LK} K_2$

$$mp_L^* = A_L + B_L L_2^* + B_{LK} K_2^*$$

<u>Capital Stock</u> $mp_K = \beta_K K_2 + \beta_{LK} L_2$

$$mp_K^* = B_{LK} L_2^*$$

[1] The common notation for Germany and France in the EEC literature are D and F for West Germany (Deutschland) and France, respectively.

CALCULATION OF CONVERSION FACTOR: ρ

Consider the following expression for the combined nominal capital stock of West Germany and France:

$$P^{87}_{80}\,\bar{K}^{87}_{80} = P^{87}_{80}\,K^{87}_{80} + P^{87}_{80}{}^{*}\,K^{87}_{80}{}^{*}\,E^{87}$$

where

P^{87}_{80} = 1987 price index for West German net capital stock (base year 1980).

$P^{87}_{80}{}^{*}$ = 1987 price index for French net capital stock (base year 1980).

E^{87} = nominal exchange rate expressed in D-Marks per French francs for all traded goods and services.

$\bar{K}^{87}_{80}$ = real net capital stock for both countries measured in 1980 base year prices.

K^{87}_{80} = West German real net capital stock measured in 1980 base year West German prices.

$K^{87}_{80}{}^{*}$ = French real net capital stock measured in 1980 base year French prices.

Dividing through both sides of the initial expression by P^{87}_{80} obtains:

$$\bar{K}^{87}_{80} = K^{87}_{80} + K^{87}_{80}{}^{*}\left(\frac{P^{87}_{80}{}^{*}}{P^{87}_{80}}\,E^{87}\right)$$

Then

$$\rho = \left(\frac{P^{87}_{80}{}^{*}}{P^{87}_{80}}\,E^{87}\right)$$

ELASTICITIES

Percentage changes in the increase of factor inputs and national output are easily obtained by comparing before and after trade marginal products and national outputs. For the home country in case two we can note:

$$\frac{f_L(L, K, T) - f_L(\bar{L}, \bar{K}, T)}{f_L(L, K, T)} \cdot 100 = \%\Delta mp_L$$

$$\frac{f_k(L, K, T) - f_k(\bar{L}, \bar{K}, T)}{f_k(L, K, T)} \cdot 100 = \%\Delta mp_L$$

Further

$$\frac{Y - Y(\bar{L}, \bar{K}, T)}{Y} \cdot 100 = \%\Delta Y$$

Similar analysis can be applied to case 1.

In addition to the above results comparative statics can be generated for factor input demands and national output. By way of example consider the following: What effect will the recent flow of immigrants to West Germany from East Germany have on the French worker once the French/German border is opened to the free flow of factor inputs? Accordingly, what effect will the flow of immigrants to Germany have on the European Community's ability to supply national product to world markets?[1]

In order to answer the first question let Y* be the national product function of France. further, let

$$mp_L^* = f_L(\bar{L}, \bar{K}, T)$$

be the marginal product of French labor before immigration, but after European integration. Now consider the effect of a 1% increase of the jointly employed labor forces of the two countries in French wages. Initially, it can be noted that:

$$mp_L^* = g_L(\bar{L}, \bar{K}, T)$$

[1]Assume in these examples that both countries represent the entire European community.

From which obtains

$$\varepsilon_{L\bar{L}} = \left(\frac{\bar{L}}{mp_L^*}\right) \cdot \left(\frac{\partial mp_L^*}{\partial \bar{L}}\right)$$

where $\varepsilon_{L\bar{L}}$ is the labor elasticity of the French marginal product of labor. French labor income can be expected to rise or fall by this amount.

In answer to the second question, we might wish to know the effect of this same wave of new immigrants on the joint production of France and Germany after European integration. After integration it can be shown that

$$Y = Y(\bar{L}, \bar{K}, T)$$

and

$$Y^* = Y^*(\bar{L}, \bar{K}, T)$$

Then

$$\varepsilon_{Y\bar{L}} = \left(\frac{\bar{L}}{Y}\right) \cdot \left(\frac{\partial Y}{\partial \bar{L}}\right)$$

$$\varepsilon_{Y^*\bar{L}} = \left(\frac{\bar{L}}{Y^*}\right) \cdot \left(\frac{\partial Y^*}{\partial \bar{L}}\right)$$

Then, a 1% increase in the employed work force of the European Community (composed only of France and West Germany in this example) would result in an overall increase in European output equal to:

$$\varepsilon_{Y\bar{L}} + \varepsilon_{Y^*\bar{L}}$$

The list of alternative questions which can be answered by this model is quite long.

FLEXIBILITY CONDITIONS

* A continuous function with first and second order differentiability in its arguents can be minimally described by the following number of equations:

$$(n + 1)^2 + (n + 1) + 1$$

where $n =$ the number of factor inputs,[1]

$(n + 1)^2 =$ the number of relevant elements in the matrix of second order partials, [2]

$(n + 1) =$ the number of first order partials,

$1 =$ the objective function (the national production function).

* Symmetry of the matrix of second order partials is an important criterium for the integrability of factor demands. It can be described by

$$\frac{(n + 1)^2 - (n + 1)}{2}$$

different equations of the general form:

$$\frac{\delta^2 y}{\delta x_1 \delta x_2} = \frac{\delta^2 y}{\delta x_2 \delta x_1} \quad .$$

where $i, j = 1, 2, \ldots, n + 1$
and $i \neq j$.

* Linear homogeneity in factor prices and national surplus (the adding-up condition) is completely described by a single equation.[3]

$$C(w, r, \pi, Y, T) = \frac{\delta C}{\delta w} \cdot w + \frac{\delta C}{\delta r} \cdot r + \frac{\delta C}{\delta \pi} \cdot \pi$$

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[1]Because neither π, nor T is crucial in the determination of our general equilibrium condtions we may ignore them here and thus reduce the size of our estimable parameter set. Y must be included in order to accommodate change in total cost when relative factor prices are constant..

[2]Because $\frac{\delta C}{\delta \pi} = 1$ and $\frac{\delta^2 C}{\delta \pi \delta y} = \frac{\delta^2 C}{\delta \pi \delta r} = \frac{\delta^2 C}{\delta \pi \delta w} = 0$ we are not conerned with differentiability of C with respect to π.

[3]National surplus is a residual term which only comes into play in the absence of linear homogeneity in the other arguments -- i.e., when constant returns to scale in variable factor inputs are not present.

where $\frac{\delta C}{\delta w} = L(w, r, p\ Y, T),$

$$\frac{\delta C}{\delta r} = K(w, r, p\ Y, T),$$

$$\frac{\delta C}{\delta \pi} = 1.$$

* Zero degree homogenity in factor demands with respect to prices yields as many equations as there are factor arguments; in this case two:

$$L = L(tw, tr, p\ Y, T)$$
$$K = K(tw, tr, p\ Y, T)$$

The condition of zero degree homogeniety does not apply to constraints. Thus, π is excluded from this set of condtions.

* Linear homogeneity of marginal cost in factor prices and output. this constitutes a singel equation:

$$\frac{\delta C}{\delta Y} = \frac{\delta^2 C}{\delta Y\,\delta w} \cdot w + \frac{\delta^2 C}{\delta Y\,\delta r} \cdot r + \frac{\delta^2 C}{\delta Y\,\delta \pi} \cdot \pi$$

by Youngs' Theorem:

$$\frac{\delta C}{\delta Y} = \frac{\delta^2 C}{\delta w\,\delta Y} \cdot w + \frac{\delta^2 C}{\delta r\,\delta Y} \cdot r$$

where $\frac{\delta^2 C}{\delta Y\,\delta \pi} = 0$

Subtracting the number of constraints from the total number of equations required to obtain first and second order differentiability yields the following number of equations:

$$(n + 1)^2 + (n + 1) + 1 - \left(\frac{(n + 1)^2 - (n + 1)}{2} + n + 1 + 1 \right) =$$

$$= \frac{(n + 2)(n + 1)}{2}$$

If the model is exactly identified, then the minimum number of estimable parameters is 6, where n = 2.

SIMPLE QUADRATIC FUNCTION

A TAYLOR SERIES EXPANSION

Consider the following expansion of the function f(L,K,T) around the point $f(L_o,K_o,T_o)$:

$$
\begin{aligned}
f(L,K,T) = \; & f(L_o, K_o, T_o) + \\
& f_L(L_o, K_o, T_o) \cdot (L - L_o) + \\
& f_K(L_o, K_o, T_o) \cdot (K - K_o) + \\
& f_K(L_o, K_o, T_o) \cdot (T - T_o) + \\
& \frac{1}{2} [\; f_{LL}(L_o, K_o, T_o) \cdot (L - L_o)^2 + \\
& f_{KK}(L_o, K_o, T_o) \cdot (K - K_o)^2 + \\
& f_{TT}(L_o, K_o, T_o) \cdot (T - T_o)^2 + \\
& 2\, f_{LT}(L_o, K_o, T_o) \cdot (L - L_o)(T - T_o) + \\
& 2\, f_{KT}(L_o, K_o, T_o) \cdot (K - K_o)(T - T_o) + \\
& 2\, f_{LK}(L_o, K_o, T_o) \cdot (L - L_o)(K - K_o) \;]
\end{aligned}
$$

Expanding terms the terms in parantheses and rearranging obtains:

$$
\begin{aligned}
f(L,K,T) = \; & (\; f^o - f^o_L L_o - f^o_K K_o - f^o_T T_o + \frac{1}{2} f^o_{LL} L_o^2 + \frac{1}{2} f^o_{KK} K_o^2 + \frac{1}{2} f^o_{TT} T_o^2 + \\
& f^o_{LT} L_o T_o + f^o_{KT} K_o T_o + f^o_{LK} L_o K_o) + \\
& (\; f^o_L - f^o_{LL} L_o - f^o_{LT} T_o - f^o_{LK} K_o)\, L + \\
& (\; f^o_K - f^o_{KK} K_o - f^o_{KT} T_o - f^o_{LK} L_o)\, K + \\
& (\; f^o_T - f^o_{TT} T_o - f^o_{LT} L_o - f^o_{KT} K_o)\, T + \\
& \frac{1}{2} f^o_{LL} L^2 + \frac{1}{2} f^o_{KK} K^2 + \frac{1}{2} f^o_{TT} T^2 + f^o_{LT} LT + f^o_{KT} KT + f^o_{LK} LK
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
Y = \; & \alpha_o + \alpha_L L + \alpha_K K + \alpha_T T + \frac{1}{2} \beta_L L^2 + \frac{1}{2} \beta_K K^2 + \frac{1}{2} \beta_T T^2 + \\
& \gamma_{LT} LT + \gamma_{KT} KT + \gamma_{LK} LK
\end{aligned}
$$

such that $\quad Y = f(L,K,T)$

$$\alpha_o = \left(f^o - f^o_L L_o - f^o_K K_o - f^o_T T_o + \frac{1}{2} f^o_{LL} L_o^2 + \frac{1}{2} f^o_{KK} K_o^2 + \frac{1}{2} f^o_{TT} T_o^2 \right.$$
$$\left. f^o_{LT} L_o T_o + f^o_{KT} K_o T_o + f^o_{LK} L_o K_o \right)$$
$$\alpha_L = f^o_L - f^o_{LL} L_o - f^o_{LT} T_o - f^o_{LK} K_o$$
$$\alpha_K = f^o_K - f^o_{KK} K_o - f^o_{KT} T_o - f^o_{LK} L_o$$
$$\alpha_T = f^o_T - f^o_{TT} T_o - f^o_{LT} L_o - f^o_{KT} K_o$$
$$\beta_L = f^o_{LL}$$
$$\beta_K = f^o_{KK}$$
$$\beta_T = f^o_{TT}$$
$$\gamma_{LT} = f^o_{LT}$$
$$\gamma_{KT} = f^o_{KT}$$
$$\gamma_{LK} = f^o_{LK}$$

Hence, a simple linear quadratic form.

FUNCTIONAL FORMS

Cholesky Transformation

The choice of an appropriate functional form is subject to a large variety of criteria. Important criteria include theoretical appropriateness, flexibility, compatibility with the data, and ease of application. In this study we will examine two versions of the simple quadratic function: linear and nonlinear. Although both of these forms are theoretically flexible, empirically flexibility can be established for neither. Notwithstanding, partial flexibility can be demonstrated for the non-linear form. Both the linear and nonlinear forms required major respecification before reliable estimates could be obtained. Although the CES and Cobb-Douglas functions are much more widely used in similar analyses,[20] the quadratic function represents a Taylor series expansion and is particularly appropriate for estimation[21]. Furthermore, the quadratic function makes the calculation of comparative statics especially easy. Formally, the linear and nonlinear versions of the statistical model are given below.

Linear Model:

(11) $$Y = a_0 + a_L L + a_K + a_T T + \frac{1}{2} b_L L^2 + \frac{1}{2} b_K K^2 + \frac{1}{2} b_T T^2 + + g_{LT} LT + g_{KT} KT + g_{LK} LK + e$$

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[20]See the introduction to a collection of works edited by. T.N. Scrinivasan and John Whalley (1986) .

[21]Taylor series expansions are useful approximations for small deviations around a single point. The wisdom of using such an approximation in the presence of a trend variable is however questionable. Fortunately for this study, the trend variable is never reliable and is eventually eliminated. Thus, the problem disappears by default.

Nonlinear Model:

(12) $$Y = a_0 + a_L^2 L + a_K^2 K + a_T T - \frac{1}{2} b_{11}^2 L^2 - \frac{1}{2}(b_{12}^2 + b_{22}^2) K^2 + + \frac{1}{2} b_T T^2 + g_{LT} L T + g_{KT} K T - b_{11} b_{12} L K + e$$

Close examination of both models reveals the following similarities:

$$a_L = a_L^2 \qquad b_L = - b_{11}^2$$

$$a_K = a_K^2 \qquad b_K = - (b_{12}^2 + b_{22}^2)$$

$$g_{LK} = - b_{11} b_{12}$$

Moving from the linear to the nonlinear model accomplishes two tasks. On the one hand, it ensures concavity in output with respect to factor inputs. On the other , it facilitates the estimation of positive marginal products. An important requisite for flexibility in cost is negative semi-definiteness. A function which is concave in its arguments is negative semi-definite in these same arguments. The production function which forms the constraint to the cost function must be concave in factor inputs. The non-linear specification of our production function ensures concavity in both cost and production.

This specification is obtained by applying the Cholesky transformation to the submatrix of second order partials and cross partials of our linear model. We apply this transformation only to observed factor inputs -- labor and capital.

Let $$A^L = \begin{pmatrix} b_L & g_{LK} & g_{LT} \\ g_{KL} & b_K & g_{KT} \\ g_{TL} & g_{TK} & b_T \end{pmatrix}$$

be the matrix of second order partials for our linear model.

Further let $$A_{LK} = \begin{pmatrix} b_L & g_{KL} \\ g_{LK} & b_K \end{pmatrix}$$

be a submatrix of A^L with respect to labor and capital only.

The Cholesky decomposition tells us that we can rewrite A_{LK} as:

$$A_{LK} = L\,D\,L' = L\,C\,C'\,L' = B\,B'$$

$$\text{where} \quad L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}; \quad D = C\,C'\,; \quad C = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{11} \end{pmatrix}$$

$$\text{and} \quad B = LC = \begin{pmatrix} L_{11}c_{11} & 0 \\ L_{21}c_{11} & L_{22}c_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$$

Finally, $$A_{LK} = B\,B' = \begin{pmatrix} b^2_{11} & b_{11}b_{12} \\ b_{11}b_{12} & b^2_{12}+b^2_{22} \end{pmatrix}$$

Multiplying by -1 we obtain

$$-A_{LK} = \begin{pmatrix} -b^2_{11} & -b_{11}b_{12} \\ -b_{11}b_{12} & -b^2_{12}+b^2_{22} \end{pmatrix}$$

Replacing the submatrix A_{LK} with our newly formulated submatrix

$-A_{LK}$ yields

$$A^{NL} = \begin{pmatrix} -b_{11}^{2} & -b_{11}b_{12} & g_{LT} \\ -b_{11}b_{12} & -\left(b_{12}^{2}+b_{22}^{2}\right) & g_{KT} \\ g_{TL} & g_{TK} & b_{T} \end{pmatrix}$$

which corresponds to the matrix of second order partial and cross-partial terms of our nonlinear production model given by equation (12).

In order for A_{LK} to be negative semi-definite it must be true that $b_L \leqq 0$, $b_K \leqq 0$ and $b_L b_K - g_{LK}^2 > 0$. Substituting b_L, b_K and g_{LK}^2 with the expressions $-b_{11}^2$, $-b_{11}b_{12}$ and $-\left(b_{12}^2 + b_{22}^2\right)$, respectively, these conditions are satisfied.

Concavity or quasiconcavity in production ensures convexity of the input set, but well-behaved factor demand functions also require that marginal products are positive, else we finish with positive marginal rates of substitution and inefficient employment of factor inputs.[22] It is for this reason that we nust respecify the coefficients of the first order terms for L and K. Consider the marginal products of labor and capital for our linear model (11).

$$mp_L = \frac{dY}{dL} = a_L + b_L L + g_{LT} T + g_{LK} K$$

and $$mp_K = \frac{dY}{dK} = a_K + b_K K + g_{KT} T + g_{LK} L$$

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[22]In effect production does take place along the postive slope of our production isoquants.

From our Cholesky transformation we know b_L and b_K can be made nonpositive. Thus, positive marginal products of labor and capital can only be obtained when

$$a_L + g_{LT} T + g_{LK} K > - b_L L > 0$$

$$a_K + g_{KT} T + g_{LK} L > - b_K K > 0$$

The cross partial terms g_{LT} and g_{KT} describe marginal products of factor inputs with respect to time. Technological advancement suggests that these will be non-negative. Substitutability of factor inputs is likely to yield non-negative g_{LK}. So long as our production function is monotonic increasing raising the level of one factor input is likely to improve the marginal productivity of the other.[23] Only the signs of the constant terms a_L and a_K are not immediately obvious. In order to insure that these are positive we redefine them as the square of themselves - namely, $a_L^2 = a_L$ and $a_K^2 = a_L$. Squaring a_L and a_K to obtain non-negative values for a_L and a_K does not guarantee that the resulting marginal products will be positive; this can only be determined by the empirical interplay of all terms taken together. It does however increase the likelihood. Notwithstanding, the unstructured nature of this modelling procedure places important limitations on its empirical usefulness. Positive marginal products for individual factor inputs can only be insured over the range of factor inputs used in the estimation[24].

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[23] In the contour plot given in graphs #9 and #10 we observe postive monotonicity in factor inputs.

[24] The very notion of a quadratic production function assumes the existence of a production maximum. Depending upon the values of the parameter estimates, and the level of factors employed, it is possible that marginal products will turn negative over given ranges of factor inputs. Fortunately, this did not occur.

Extrapolation into the future using simulated forecasting techniques, and post-trade calculations of equilibrium input levels will likely extend beyond the range of observed inputs and may result in negative values for these functions. Although positive marginal products were obtained for all years, negative output was recorded for Germany under scenario II.

MODEL MECHANICS

After we have selected an appropriate functional form and have obtained reliable parameter estimates, we will wish to determine marginal products, the direction of factor movements, changes in relative factor incomes and national products. The mechanics of these measures are discussed in this section.

Taking the first order partials of each production function yields the following sets of marginal products:

From $$Y = f(L,K,T)$$

we obtain
$$mp_L = f_L(L, K, T)$$
$$mp_K = f_K(L, K, T).$$

From $$Y^* = g(L^*,K^*,T)$$

we obtain
$$mp^*_L = g_L(L^*, K^*, T)$$
$$mp^*_K = g_K(L^*, K^*, T).$$

From these we can examine either of two trade scenarios: one, country specific capital stock two, country non-specific capital stock.

Scenario 1

Country specific capital stock implies immobility. In some ways this is the less realistic of the two scenarios; generally it is easier to transport equipment than to transport people. Also, country specific capital stock emphasizes capital structures and fails thereby to model the mobile and homogeneous nature of capital equipment.

Geographically Germany and France share long contiguous borders. In the absence of rigid customs regulations French and German citizens may commute easily across national boundaries in search of employment and monthly household income. This is already standard practice for some. In order to capture both the direction and magnitude of this movement we set the marginal physical products of labor (the competitive real wage rate) in each country equal and allow the free flow of labor services between countries until this expression is satisfied. Formally, we write[1]:

(5a) $$f_L(L,K,T) = g_L(L^*,K,^*,T) \cdot r_Y$$

such that
$$\bar{L} = L + L^*$$
$$K = K$$
$$K^* = K^*$$

where K = home country fixed capital structures and equipment.

K^* = foreign country fixed capital structures and equipment.

$r_Y = \dfrac{P^*_Y}{P_Y} E$ = Currency conversion factor (national output).

P^*_Y = Price index for national output (Home Country).

P_Y = Price index for national output (Foreign Country).

Substituting our full-employment condition into expression (5) yields the following relationship:

(5b) $$f_L(L, K, T) = g_L((\bar{L}\text{-}L), K^*, T) \cdot r_Y$$

From this expression and our full employment condition we can solve for home and foreign country equilibrium labor inputs.

$$L_1 = L_1(\bar{L}, K, K^*, r_Y, T)$$

and $$L^*_1 = L^*_1(\bar{L}, K, K^*, r_Y, T) = \bar{L} - L_1(\bar{L}, K, K^*, r_Y, T)$$

Furthermore, we can calculate the after-trade national output for both countries by substituting our post-trade equilibrium values for labor into our original production functions to obtain:

$$Y_1 = Y_1(\bar{L}, K, K^*, r_Y, T) = f(L_1(\bar{L}, K, K^*, r_Y, T), K, T)$$

---

[1]Whereas $mp_L = mp^*_L \cdot \rho_Y$ in equilibrium, generally $mp_K \neq mp^*_K \cdot \rho_Y$.

and $$Y_1^* = Y_1^*(\bar{L}, K, K^*, r_Y, T) = g(L_1^*(\bar{L}, K, K^*, r_Y, T), K^*, T)$$

Once the barrier to factor movements s removed individual nations must coordinate their immigration and foreign investment policies vis-à-vis nonmember nations. For example the recent immigration of East Germans to West Germany would affect the size of the employed labor pool in both West Germany and France. This in turn would affect marginal physical products of both capital and labor, as well as both countries' national products. Similarly, French immigration policy toward North Africans would have a direct affect on Germany's overall economic performance. Independent action on the part of either government with respect to foreign investment would also spill over into the economic behavior of the other country. These effects are captured by the following sets of comparative statics.

<u>Comparative Statics</u> (Scenario 1):

From our equilibrium condition 5b we know:

$$mp_L = mp_L^* \cdot r_Y$$

Substituting our equilibrium demand functions $L_1(\bar{L}, K, K^*, r_Y, T)$ and $L_1^*(\bar{L}, K, K^*, r_Y, T)$ we obtain the following identity:

(6) $$f_L(L_1(\bar{L}, K, K^*, r_Y, T), K, T) \equiv g_L((\bar{L} - L_1(\bar{L}, K, K^*, r_Y, T)), K^*, T) \cdot r_Y$$

Differentiating with respect to $\bar{L}$, K and K* we obtain our comparative statics for home and foreign country labor demand.

$\bar{L}$: From above $$f_{LL}\frac{dL_1}{d\bar{L}} = g_{LL}\left(1 - \frac{dL_1}{d\bar{L}}\right) r_Y$$

we obtain $$\frac{dL_1}{d\bar{L}} = \frac{g_{LL} r_Y}{f_{LL} + g_{LL} r_Y}$$

From our full-employment condition

$$L_1^* = \bar{L} - L_1(\bar{L}, K, K^*, r_Y, T)$$

we obtain upon differentiation:

$$\frac{dL_1^*}{d\bar{L}} = 1 - \frac{dL_1}{d\bar{L}}$$

Substituting for $\frac{dL_1}{d\bar{L}}$ from above we have:

$$\frac{dL_1^*}{d\bar{L}} = \frac{f_{LL}}{f_{LL} + g_{LL} r_Y}$$

K: From (6) $$f_{LL}\frac{dL_1}{dK} + f_{LK} = g_{LL}\left(-\frac{dL_1}{dK}\right) r_Y$$

we obtain $$\frac{dL_1}{dK} = \frac{-f_{LK}}{f_{LL} + g_{LL} r_Y}$$

From our full employment condition

$$L_1^* = \bar{L} - L_1(\bar{L}, K, K^*, r_Y, T)$$

Differentiating and substituting yields

$$\frac{dL_1^*}{dK} = -\frac{dL_1}{dK} = \frac{f_{LK}}{f_{LL} + g_{LL} r_Y}$$

K*: From (6) $$f_{LL}\frac{dL_1}{dK^*} = \left(g_{LL}\left(-\frac{dL_1}{dK^*}\right) + g_{LK}\right) r_Y$$

we obtain $$\frac{dL_1}{dK^*} = \frac{g_{LK} r_Y}{f_{LL} + g_{LL} r_Y}$$

From our full employment condition

$$L_1^* = \bar{L} - L_1(\bar{L}, K, K^*, r_Y, T)$$

Differentiating and substituting yields

$$\frac{dL_1^*}{dK^*} = -\frac{dL_1}{dK^*} = \frac{-g_{LK}r_Y}{f_{LL} + g_{LL}r_Y}$$

Substituting our demand functions into our national production functions we can determine the effects of changes in $\bar{L}$, K and K* on national outputs. Differentiating

$$Y_1 = f(L_1(\bar{L}, K, K^*, r_Y, T), K, T)$$

$$Y_1^* = g((\bar{L} - L_1(\bar{L}, K, K^*, r_Y, T)), K, T)$$

with respect to $\bar{L}$, K and K* we obtain:

$\bar{L}$:

$$\frac{dY_1}{d\bar{L}} = f_L\frac{dL_1}{d\bar{L}} = f_L\left(\frac{g_{LL}r_Y}{f_{LL} + g_{LL}r_Y}\right) = \frac{f_L g_{LL}r_Y}{f_{LL} + g_{LL}r_Y}$$

and

$$\frac{dY_1^*}{d\bar{L}} = g_L\left(1 - \frac{dL_1}{d\bar{L}}\right) = g_L\left(\frac{f_{LL}}{f_{LL} + g_{LL}r_Y}\right) = \frac{g_L f_{LL}}{f_{LL} + g_{LL}r_Y}$$

K: Similarly

$$\frac{dY_1}{dK} = f_L\frac{dL_1}{dK} + f_K = f_L\left(\frac{-f_{LK}}{f_{LL} + g_{LL}r_Y}\right) + f_K$$

$$= \frac{f_K(f_{LL} + g_{LL}r_Y) - f_L f_{LK}}{f_{LL} + g_{LL}r_Y}$$

and

$$\frac{dY_1^*}{dK} = g_L\left(-\frac{dL_1}{dK}\right) = \frac{g_L f_{LK}}{f_{LL} + g_{LL}r_Y}$$

K*: and finally

$$\frac{dY_1}{dK^*} = f_L \frac{dL_1}{dK^*} = f_L \frac{g_{LK} r_Y}{f_{LL} + g_{LL} r_Y} = \frac{f_L g_{LK} r_Y}{f_{LL} + g_{LL} r_Y}$$

and

$$\frac{dY_1^*}{dK^*} = g_L \left(\frac{dL_1}{dK^*} \right) + g_K = g_L \left(\frac{g_{LK} r_Y}{f_{LL} + g_{LL} r_Y} \right) + g_K$$

$$= \frac{g_K (f_{LL} + g_{LL} r_Y) - g_L g_{LK} r_Y}{f_{LL} + g_{LL} r_Y}$$

Of still further interest are the effects of national resource endowments on factor prices and factor incomes. Substituting equilibrium factor demands into our marginal product functions and differentiating with respect to $\bar{L}$, K and K*, we obtain comparative statics for equilibrium factor prices with respect to national resources K and K* and internationally shared resources $\bar{L}$. From equation (6) we obtain upon differentiating with respect to $\bar{L}$, K, and K*

$\bar{L}$: $$\frac{d}{d\bar{L}} \left(mp_{L_1} \right) = \frac{d}{d\bar{L}} \left(mp^*_{L_1} \right) = \frac{f_{LL} \; g_{LL} r_Y}{f_{LL} + g_{LL} r_Y}$$

K: $$\frac{d}{dK} \left(mp_{L_1} \right) = \frac{d}{dK} \left(mp^*_{L_1} \right) = \frac{f_{LK} \; g_{LL} r_Y}{f_{LL} + g_{LL} r_Y}$$

K*: $$\frac{d}{dK^*} \left(mp_{L_1} \right) = \frac{d}{dK^*} \left(mp^*_{L_1} \right) = \frac{f_{LL} \; g_{LK} r_Y}{f_{LL} + g_{LL} r_Y}$$

Because capital stock is specific for each country, the marginal products of capital stock will generally be different for each country. Thus we would also expect the corresponding comparative statics to be different. Differentiating our post-trade marginal products for capital stock for each country with respect to $\bar{L}$, K, and K* we obtain:

Germany

$$\frac{d}{d\bar{L}}\left(mp_{K1}\right) = \frac{f_{KL}\,g_{LL}\,r_Y}{f_{LL} + g_{LL}\,r_Y}$$

$$\frac{d}{dK}\left(mp_{K1}\right) = \frac{f_{KL}\,f_{KK} - f_{LK}^{2} + f_{KK}\,g_{LL}\,r_Y}{f_{LL} + g_{LL}\,r_Y}$$

$$\frac{d}{dK^*}\left(mp_{K1}\right) = \frac{f_{LK}\,g_{LK}\,r_Y}{f_{LL} + g_{LL}\,r_Y}$$

France

$$\frac{d}{d\bar{L}}\left(mp^{*}_{K1}\right) = \frac{f_{KL}\,g_{LL}\,r_Y}{f_{LL} + g_{LL}\,r_Y}$$

$$\frac{d}{dK}\left(mp^{*}_{K1}\right) = \frac{f_{LK}\,g_{LL}\,r_Y}{f_{LL} + g_{LL}\,r_Y}$$

$$\frac{d}{dK^*}\left(mp_{K1}\right) = \frac{f_{LL}\,g_{KK} + \left(g_{LL}\,g_{KK} - g_{LK}\right)r_Y}{f_{LL} + g_{LL}\,r_Y}$$

<u>Scenario 2</u>

In this scenario we allow the free flow of <u>both</u> labor and capital. In order to obtain equilibrium factor demands we set the marginal physical products of each country equal, and subject these to our full employment constraints. Formally,.

(7a) $$f_L(L, K, T) = g_L(L^*, K^*, T) \cdot r_Y$$

(8a) $$f_K(L, K, T) = g_K(L^*, K^*, T) \cdot r_Y$$

$$\bar{L} = L + L^*$$

$$\bar{K} = K + K^* \cdot r_K$$

Substituting for L* and K* from our full employment constraints the above system simplifies to two equations in two unknowns, L and K:

(7b) $$f_L(L,K,T) = g_L\left(\bar{L}-L,\ (\bar{K}-K)\,\frac{1}{r_K},\ T\right) \cdot r_Y$$

(8b) $$f_K(L,K,T) = g_K(\bar{L}-L,\ (\bar{K}-K)\frac{1}{r_K}, T)\cdot r_Y$$

Solving for L and K we obtain

$$L_2 = L_2(\bar{L}, \bar{K}, r_Y, r_K, T)$$

$$K_2 = K_2(\bar{L}, \bar{K}, r_Y, r_K, T)$$

and $$L^*_2 = L^*_2(\bar{L}, \bar{K}, r_Y, r_K, T) = \bar{L} - L_2(\bar{L}, \bar{K}, r_Y, r_K, T)$$

$$K^*_2 = K^*_2(\bar{L}, \bar{K}, r_Y, r_K, T) = \left(\bar{K} - K_2(\bar{L}, \bar{K}, r_Y, r_K, T)\right)\cdot\frac{1}{r_K}$$

Notice that factor demands are no longer determined by the country specific capital stocks, K and K*; rather they are determined by the internationally shared capital stock, $\bar{K}$.

Substituting back into our original production function we obtain the after-trade levels of national output given by:

$$Y_2 = Y_2(\bar{L}, \bar{K}, r_Y, r_K, T)$$

and $$Y^*_2 = Y^*_2(\bar{L}, \bar{K}, r_Y, r_K, T)$$

When both factors are traded the corresponding marginal products for each country will be equal. This allows us to write the following set of identities:

(7c) $$f_L\ (L_2(\bar{L}, \bar{K}, r_Y, r_K, T), K_2(\bar{L}, \bar{K}, r_Y, r_K, T), T) \equiv$$

$$g_L((\bar{L} - L_2(\bar{L}, \bar{K}, r_Y, r_K, T)), (\bar{K} - K_2(\bar{L}, \bar{K}, r_Y, r_K, T))\frac{1}{r_K}, T)\cdot r_Y$$

(8c) $$f_K(L_2(\bar{L}, \bar{K}, r_Y, r_K, T), K_2(\bar{L}, \bar{K}, r_Y, r_K, T), T) \equiv$$

$$g_K((\bar{L} - L_2(\bar{L}, \bar{K}, r_Y, r_K, T)), (\bar{K} - K_2(\bar{L}, \bar{K}, r_Y, r_K, T))\frac{1}{r_K}, T) \cdot r_Y$$

<u>Comparative Statics</u> (Scenario 2):

The comparative statics corresponding to the above identities are given below.

Differentiating 7c and 8c with respect to $\bar{L}$ and applying Young's theorem we obtain

$$f_{LL}\frac{dL_2}{d\bar{L}} + f_{LK}\frac{dK_2}{d\bar{L}} = g_{LL}\left(1 - \frac{dL_2}{d\bar{L}}\right) \cdot r_Y - g_{LK}\frac{dK_2}{d\bar{L}} \cdot \frac{r_Y}{r_K}$$

and $$f_{LK}\frac{dL_2}{d\bar{L}} + f_{KK}\frac{dK_2}{d\bar{L}} = g_{LK}\left(1 - \frac{dL_2}{d\bar{L}}\right) \cdot r_Y - g_{KK}\frac{dK_2}{d\bar{L}} \cdot \frac{r_Y}{r_K}$$

Expanding and rearranging terms we obtain the following system of equations:

$$\begin{pmatrix} f_{LL} + g_{LL}r & f_{LK} + g_{LK}\frac{r_Y}{r_K} \\ f_{LK} + g_{LK}r & f_{KK} + g_{KK}\frac{r_Y}{r_K} \end{pmatrix} \begin{pmatrix} \frac{dL_2}{d\bar{L}} \\ \frac{dK_2}{d\bar{L}} \end{pmatrix} = \begin{pmatrix} g_{LL} \\ g_{LK} \end{pmatrix} \cdot r_Y$$

In matrix algebra we have Ax = b. Applying Cramer's rule we obtain

(9a) $$\frac{dL_2}{d\bar{L}} = \frac{|A_1|}{|A|}$$

(9b) $$\frac{dK_2}{d\bar{L}} = \frac{|A_2|}{|A|}$$

where $$|A_1| = g_{LL}r_Y\left(f_{KK} + g_{KK}\frac{r_Y}{r_K}\right) - g_{LK}r_Y\left(f_{LK} + g_{LK}\frac{r_Y}{r_K}\right)$$

$$|A_2| = g_{LK}r_Y\left(f_{LL} + g_{LL}r_Y\right) - g_{LL}r_Y\left(f_{LK} + g_{LK}r_K\right)$$

$$|A| = \left(f_{LL} + g_{LL} r_Y\right)\left(f_{KK} + g_{KK}\frac{r_Y}{r_K}\right) - \left(f_{LK} + g_{LK} r_Y\right)\left(f_{LK} + g_{LK}\frac{r_Y}{r_K}\right)$$

Differentiating our identities with respect to $\bar{K}$ and applying Young's theorem we obtain

$$f_{LK}\frac{dL_2}{d\bar{K}} + f_{LK}\frac{dK_2}{d\bar{K}} = -g_{LL}\frac{dL_2}{d\bar{K}}\cdot r_Y + g_{LK}\frac{r_Y}{r_K} - g_{LK}\frac{dK_2}{d\bar{K}}\cdot\frac{r_Y}{r_K}$$

and

$$f_{LK}\frac{dL_2}{d\bar{K}} + f_{KK}\frac{dK_2}{d\bar{K}} = -g_{LK}\frac{dL_2}{d\bar{K}}\cdot r_Y + g_{KK}\frac{r_Y}{r_K} - g_{KK}\frac{dK_2}{d\bar{K}}\cdot\frac{r_Y}{r_K}$$

Expanding and rearranging terms yields the following system of equations:

$$\begin{pmatrix} f_{LL} + g_{LL} r & f_{LK} + g_{LK}\frac{r_Y}{r_K} \\ f_{LK} + g_{LK} r & f_{KK} + g_{KK}\frac{r_Y}{r_K} \end{pmatrix}\begin{pmatrix} \frac{dL_2}{d\bar{K}} \\ \frac{dK_2}{d\bar{K}} \end{pmatrix} = \begin{pmatrix} g_{LL} \\ g_{LK} \end{pmatrix}\cdot\frac{r_Y}{r_K}$$

Once again we have a system of two equations in two unknowns which is of the general matrix form Ax = b. Applying Cramer's rule we solve for x to obtain.

(10a) $$\frac{dL_2}{d\bar{K}} = \frac{|A_1|}{|A|}$$

(10b) $$\frac{dK_2}{d\bar{K}} = \frac{|A_2|}{|A|}$$

where $$|A_1| = g_{LK}\frac{r_Y}{r_K}\left(f_{KK} + g_{KK}\frac{r_Y}{r_K}\right) - g_{KK}\frac{r_Y}{r_K}\left(f_{LK} + g_{LK}\frac{r_Y}{r_K}\right)$$

$$|A_2| = g_{KK}\frac{r_Y}{r_K}\left(f_{LL} + g_{LL}\, r_Y\right) - g_{LK}\frac{r_Y}{r_K}\left(f_{LK} + g_{LK}\, r_Y\right)$$

$$|A| = \left(f_{LL} + g_{LL}\, r_Y\right)\left(f_{KK} + g_{KK}\frac{r_Y}{r_K}\right)$$
$$- \left(f_{LK} + g_{LL}\, r_Y\right)\left(f_{LK} + g_{LK}\frac{r_Y}{r_K}\right)$$

Having derived our comparative statics for factor demands we can now obtain those for national and factor incomes. Substituting our factor demand functions into our national output functions we obtain

$$Y_2 = Y_2(\bar{L}, \bar{K}, r_Y, r_K, T) = f(L_2(\bar{L}, \bar{K}, r_Y, r_K, T), K_2(\bar{L}, \bar{K}, r_Y, r_K, T), T)$$

$$Y_2^* = Y_2^*(\bar{L}, \bar{K}, r_Y, r_K, T)$$
$$= g((\bar{L} - L_2(\bar{L}, \bar{K}, r_Y, r_K, T)), (\bar{K} - K_2(\bar{L}, \bar{K}, r_Y, r_K, T)), T)$$

Differentiating with respect to $\bar{L}$ and $\bar{K}$ we obtain

Germany $$\frac{dY_2}{d\bar{L}} = f_L(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2}{d\bar{L}} + f_K(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dK_2}{d\bar{L}}$$

$$\frac{dY_2}{d\bar{K}} = f_L(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2}{d\bar{K}} + f_K(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dK_2}{d\bar{K}}$$

France $$\frac{dY_2^*}{d\bar{L}} = f_L(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2^*}{d\bar{L}} + f_K(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dK_2^*}{d\bar{L}}$$

$$\frac{dY_2^*}{d\bar{K}} = f_L(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2^*}{d\bar{K}} + f_K(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dK_2^*}{d\bar{K}}$$

We may then substitute for $\frac{dL_2^*}{d\bar{L}}$, $\frac{dK_2^*}{d\bar{L}}$, $\frac{dL_2^*}{d\bar{K}}$, and $\frac{dK_2^*}{d\bar{K}}$ from above.[2] The effect of changes in pooled resources on factor prices is given according to the following:

Differentiating our factor price equilibrium identities 7c and 8c with respect to $\bar{L}$ and $\bar{K}$ we derive the following:

$$\frac{d}{d\bar{L}}\left(mp_{L_2}\right) = \frac{d}{d\bar{L}}\left(mp^*_{L_2}\right) \cdot r_Y$$

$$= f_{LL}(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2}{d\bar{L}} + f_{LK}(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dK_2}{d\bar{L}}$$

$$\frac{d}{d\bar{K}}\left(mp_{L_2}\right) = \frac{d}{d\bar{K}}\left(mp^*_{L_2}\right) \cdot r_Y$$

$$= f_{LK}(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2}{d\bar{K}} + f_{KK}(\bar{L}, \bar{K}, r_Y, r_K, T)\frac{dL_2}{d\bar{K}}$$

The numerical values for these relationships are given in section V for the year 1987.

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[2]Numerical values for these expressions have been calculated and listed in the results.